QUADRILATERALS

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Contents

Introduction

- A quadrilateral is a closed figure which has four sides, four angles and four vertices.
- For example, in Fig. 1.1, ABCD is a quadrilateral which has:

  (i) Four sides: AB, BC, CD, DA

  (ii) Four angles: $\angle A$, $\angle B$, $\angle C$ and $\angle D$

  (iii) Four vertices: A, B, C, D

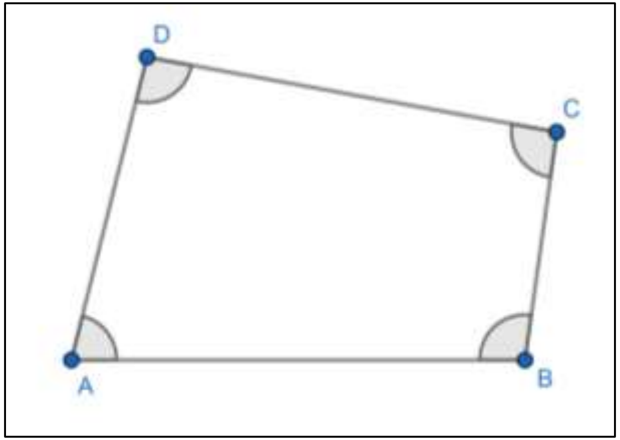

Fig. 1.1

Angle Sum Property of a Quadrilateral

- Sum of angles of a quadrilateral is 360°.

In Fig. 1.2, ABCD is a quadrilateral and AC is a diagonal.
Using Angle Sum Property of a Triangle,
In $\triangle ABC$, $\angle CAB+\angle ACB+\angle B=180^\circ$ (Eq. 1)
In $\triangle ADC$, $\angle CAD+\angle ACD+\angle D=180^\circ$ (Eq. 2)
Adding Eq. 1 and Eq. 2, we get
$(\angle CAB+\angle CAD)+(\angle ACB+\angle ACD)+\angle B+\angle D=180^\circ+180^\circ$
$\Rightarrow \angle A+\angle C+\angle B+\angle D=360^\circ$
Hence proved that sum of angles of a quadrilateral is 360°.

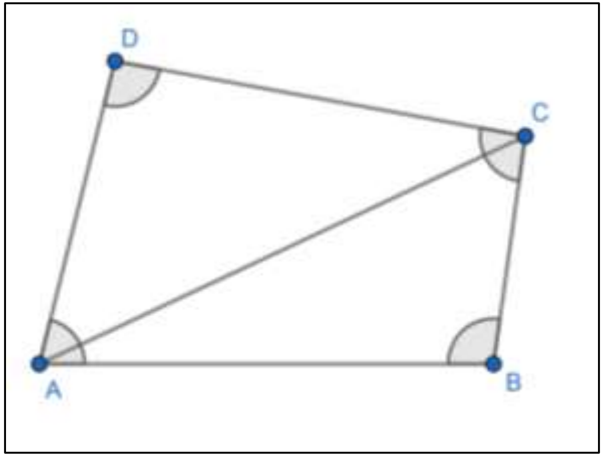

Fig. 1.2

Types of Quadrilateral

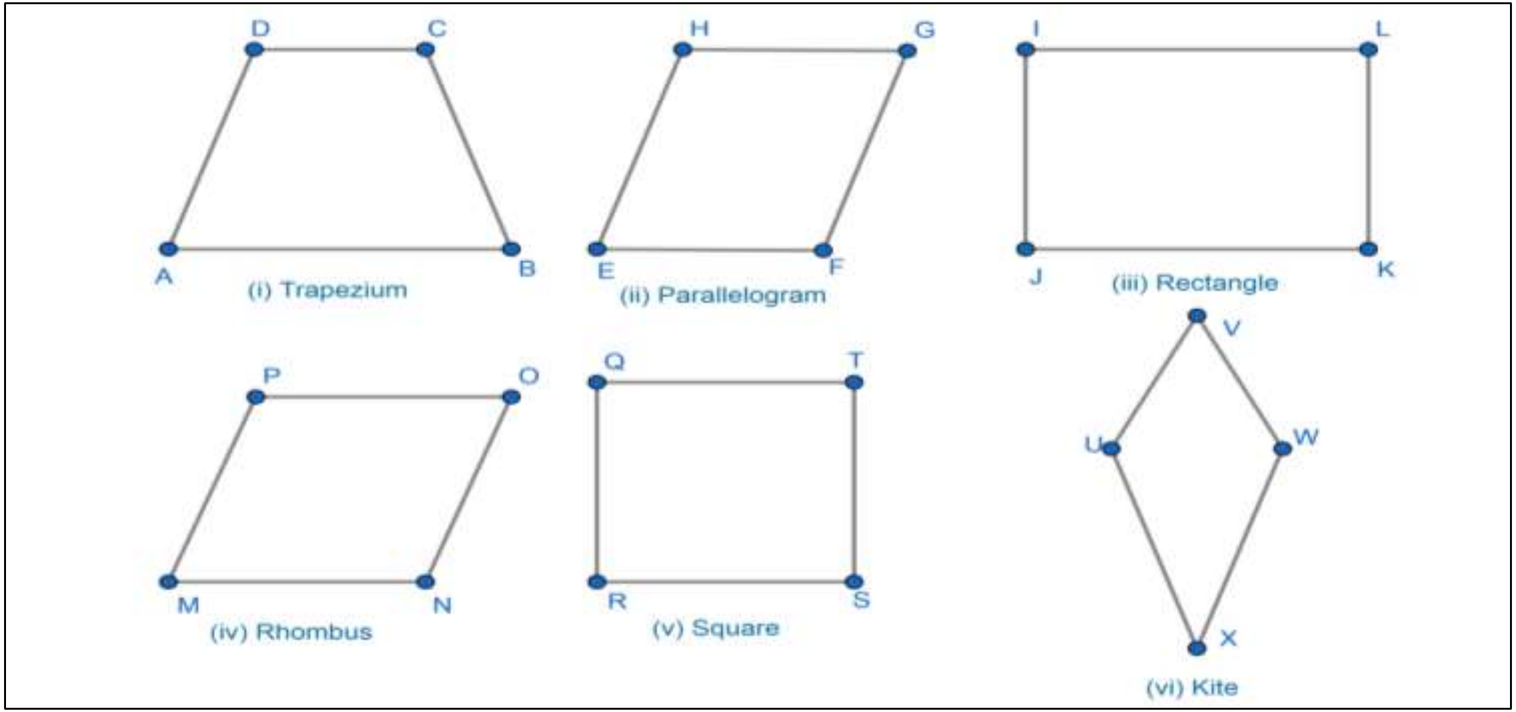

Fig. 1.3

Types of Quadrilateral (Contd..)

Observe that in Fig. 1.3 (on previous page):

i. One pair of opposite sides of quadrilateral **ABCD** is parallel, it is called **trapezium**.

ii. Both pairs of opposite sides of quadrilateral **EFGH** are parallel, it is called **parallelogram**.

iii. In quadrilateral **IJKL**, both pairs of opposite sides are parallel and all four angles are of 90°, it is called **rectangle**.

iv. In quadrilateral **MNOP**, both pairs of opposite sides are parallel and all four sides are equal, it is called **rhombus**.

v. In quadrilateral **QRST**, both pairs of opposite sides are parallel, all four sides are equal and all four angles are of 90°, it is called **square**.

vi. In quadrilateral **UVWX**, two pairs of adjacent angles are equal, it is called **kite**.

Properties of a Parallelogram

- **Theorem 1**: A diagonal of a parallelogram divides it into two congruent triangles.

Proof: In Fig. 1.4, ABCD is a parallelogram and AC is a diagonal.
In $\triangle ABC$ and $\triangle ACD$, $BC \parallel AD$ and AC is a transversal
So, $\angle BCA = \angle DAC$ (Pair of alternate angles)
Also, $AB \parallel DC$ and AC is a transversal.
So, $\angle BAC = \angle DCA$ (Pair of alternate angles)
and $AC = CA$ (Common side)
So, $\triangle ABC \cong \triangle ACD$ (By ASA Congruence rule)
or, we can say that diagonal AC divides parallelogram ABCD into two congruent triangles ABC and CDA.

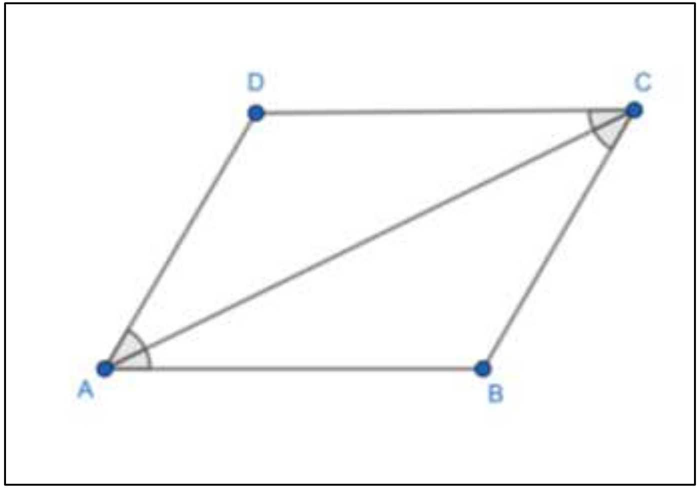

Fig. 1.4

Properties of a Parallelogram (Contd..)

- **Theorem 2**: In a parallelogram, opposite sides are equal.

Proof: In Fig. 1.5, ABCD is a parallelogram and AC is a diagonal.
In $\triangle ABC$ and $\triangle ACD$, $BC \| AD$ and AC is a transversal
So, $\angle BCA = \angle DAC$ (Pair of alternate angles)
Also, $AB \| DC$ and AC is a transversal.
So, $\angle BAC = \angle DCA$ (Pair of alternate angles)
and $AC = CA$ (Common side)
So, $\triangle ABC \cong \triangle ACD$ (By ASA Congruence rule)
Thus, $AB=CD$ (CPCT)
and $BC=AD$ (CPCT)

Therefore, in a parallelogram, opposite sides are equal.

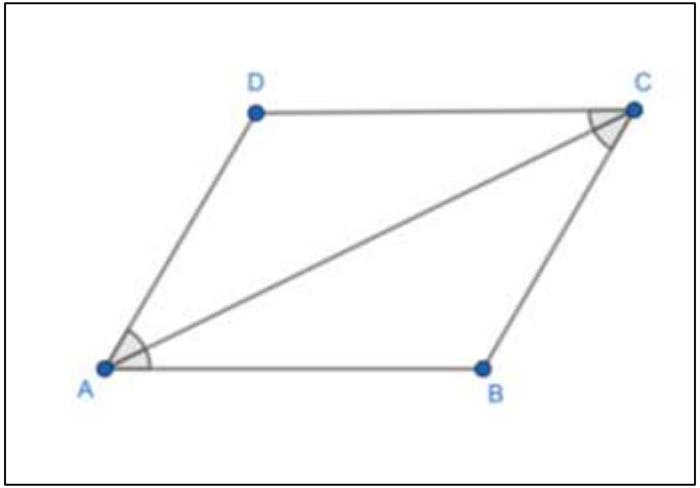

Fig. 1.5

Properties of a Parallelogram (Contd..)

- **Theorem 3**: If each pair of opposite sides of a quadrilateral is equal then it is a parallelogram

Proof: In Fig. 1.6, ABCD is a quadrilateral and AC is a diagonal. Its opposite sides are equal.
Now, in $\triangle ABC$ and $\triangle ACD$,
$AB = CD$ (Since opposite sides are equal)
$BC = CD$ (Since opposite sides are equal)
$AC = CA$ (Common Side)
Thus, $\triangle ABC \cong \triangle ACD$ (By SSS Congruence Rule)
So, $\angle BAC = \angle ACD$ (CPCT)
and $\angle CAD = \angle ACB$ (CPCT)
So, $AB \parallel CD$ and $AD \parallel BC$ (Since if alternate interior angles are equal, lines are parallel)
Therefore, ABCD is a parallelogram.

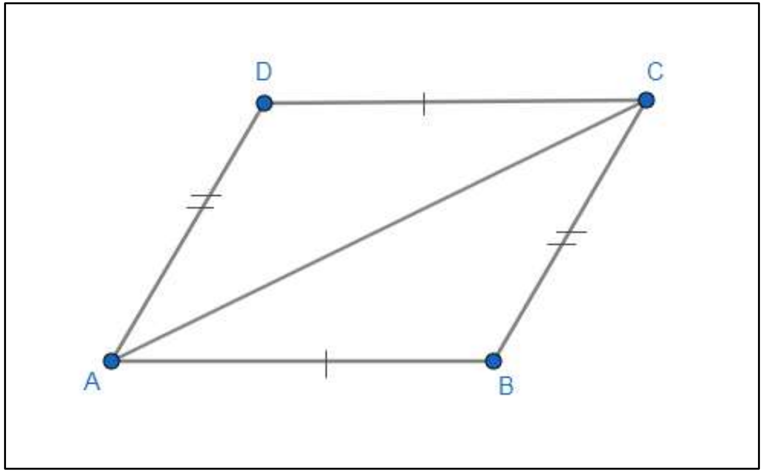

Fig. 1.6

Properties of a Parallelogram (Contd..)

- **Theorem 4**: In a parallelogram, opposite angles are equal.

Proof: In Fig. 1.7, ABCD is a parallelogram and AC is a diagonal.
Now, in $\triangle ABC \cong \triangle ACD$ (Since a diagonal of a parallelogram divides it into two congruent triangles)
So, $\angle D = \angle B$ (CPCT)
$\angle ACD = \angle BAC$ ($AB \parallel CD$ and AC is transversal, alternate interior angles are equal)
$\angle DAC = \angle ACB$ ($BC \parallel AD$ and AC is transversal, alternate interior angles are equal)
Adding above two equations, we get
$\angle ACD + \angle ACB = \angle BAC + \angle DAC$
Thus, $\angle A = \angle C$. Hence above theorem is proved.

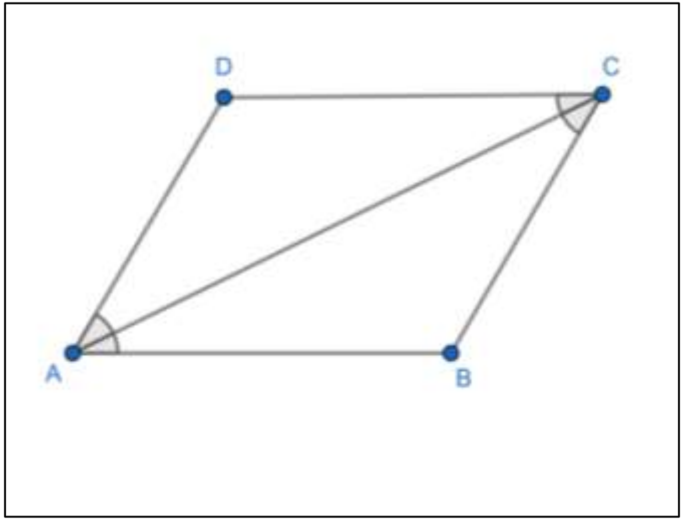

Fig. 1.7

Properties of a Parallelogram (Contd..)

- **Theorem 5**: If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.

Proof: In Fig. 1.8, ABCD is a quadrilateral and $\angle A = \angle C$ and $\angle B = \angle D$.
Now, using Angle Sum Property of a Quadrilateral, we know that:
$\angle A + \angle B + \angle C + \angle D = 360°$
Replacing $\angle C$ with $\angle A$ and $\angle D$ with $\angle B$, we get
$\angle A + \angle B + \angle A + \angle B = 360°$
$\Rightarrow 2(\angle A + \angle B) = 360°$
$\Rightarrow \angle A + \angle B = 180°$
Now AD and BC are two lines and AB is the transversal and angles on same side of transversal are supplementary, so $AD \parallel BC$.
Similarly we can prove that $AB \parallel CD$.
Thus , ABCD is a parallelogram.

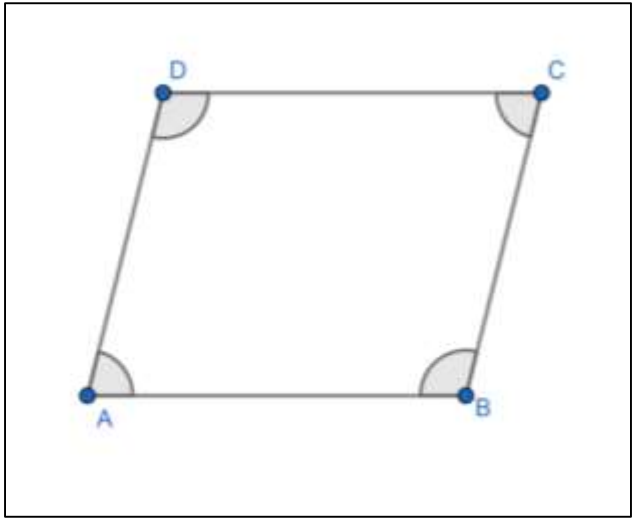

Fig. 1.8

Properties of a Parallelogram (Contd..)

- **Theorem 6**: The diagonals of a parallelogram bisect each other.

Proof: In Fig. 1.9, ABCD is a parallelogram.
Now, in $\triangle AOD$ and $\triangle BOC$,
$\angle DAC = \angle ACB$ (AD$\|$BC and AC is the transversal, so alternate interior angles are equal)
$AD = BC$ (In a parallelogram, opposite sides are equal)
$\angle ADC = \angle DBC$ (AD$\|$BC and BD is the transversal, so alternate interior angles are equal)
Thus, $\triangle AOD \cong \triangle BOC$ (By ASA Congruence Rule)
So, $OB = OD$ and $OA = OC$ (CPCT)
Hence, it is proved that diagonals of a parallelogram bisect each other.

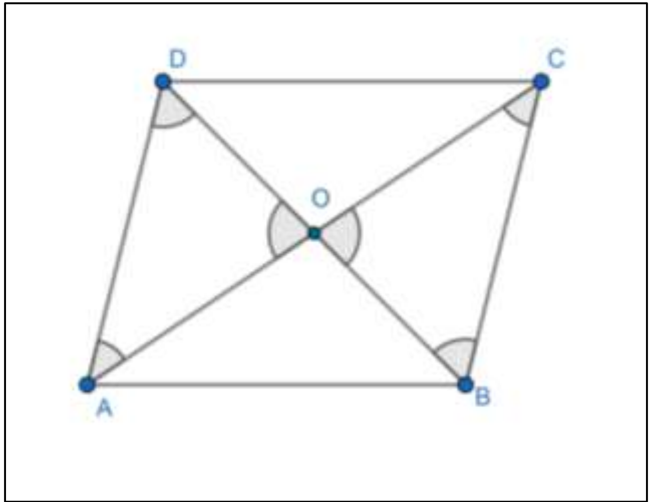

Fig. 1.9

Properties of a Parallelogram (Contd..)

- **Theorem 7**: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Proof: In Fig. 1.10, ABCD is a quadrilateral in which its diagonals bisect each other i.e. OA=OC and OB=OD.
Now, in $\triangle AOD$ and $\triangle BOC$,
OA=OC (Since the diagonals bisect each other)
$\angle AOD=\angle COB$ (Vertically Opposite Angles)
OA=OD (Since the diagonals bisect each other)
Thus, $\triangle AOD \cong \triangle BOC$ (By SAS Congruence Rule)
So, $\angle DAC=\angle ACB$ (CPCT)
Now, AD and BC are two lines and AC is the transversal and alternate interior angles are equal.
So, $AD \parallel BC$. Similarly we can prove that $AB \parallel CD$ by proving $\triangle AOB \cong \triangle DOC$ by SAS Congruence Rule.
Hence, it is proved that ABCD is a parallelogram since opposite pairs of sides are parallel.

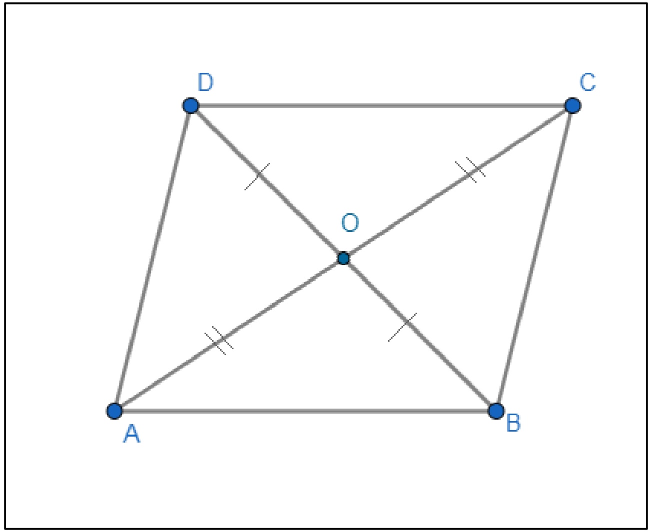

Fig. 1.10

Another Condition for a Quadrilateral to be Parallelogram

- **Theorem 8**: A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel

Proof: In Fig. 1.11, ABCD is a quadrilateral in which one pair of opposite sides is equal and parallel i.e. AB=CD and AB||CD.
Now, in △ABC and △ADC,
AB=CD (Since a pair of opposite sides is equal)
∠BAC=∠ACD (Since AB||CD and AC is transversal, so alternate interior angles are equal)
AC=AC (Common Side)
Thus, △ABC≅△ACD (By SAS Congruence Rule)
So, ∠BAC=∠ACD (CPCT)
Now, AB and CD are two lines and AC is the transversal and alternate interior angles are equal.
So, AB||CD. Similarly we can prove that AD||BC.
Hence, it is proved that ABCD is a parallelogram since opposite pairs of sides are parallel.

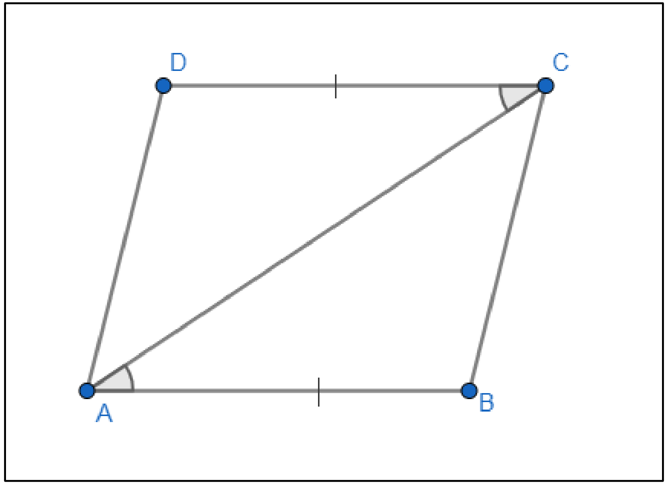

Fig. 1.11

The Mid-point Theorem

- **Theorem 9**: The line segment joining the mid-points of two sides of a triangle is parallel to the third side.

Proof: In Fig. 1.12, ABC is a triangle and E and F are mid points of AB and AC respectively. Now, EF is extended to ED and AB∥CD.
Now in △AEF and △CDF,
∠BAC=∠ACD (AB∥CD and AC is transversal, so alternate interior angles are equal)
AF=FC (Since F is mid-point of AC)
∠AFE=∠CFD (Vertically Opposite Angles)

So, △AEF≅△CDF (ASA Congruence Rule)
So, EF=DF and AE=CD=BE (CPCT)
Thus, BCDE is a parallelogram (Because one pair of opposite sides i.e. BE and CD is equal and parallel)
So, EF∥BC. Hence, above theorem is proved.

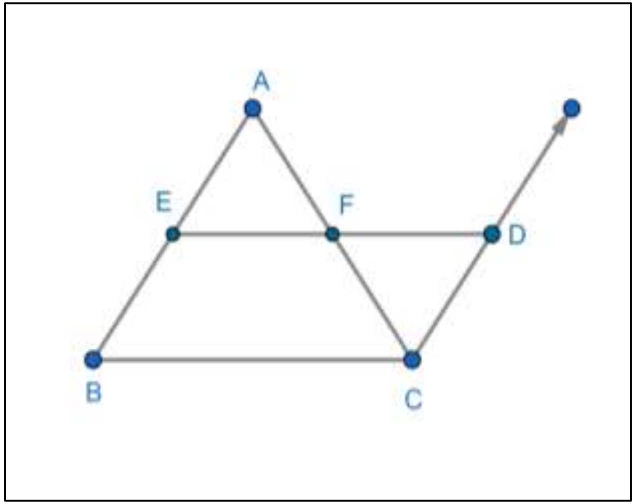

Fig. 1.12

The Mid-point Theorem (Contd..)

- **Theorem 10**: The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.

Proof: In Fig. 1.13, ABC is a triangle and E is the mid point of AB and EF∥BC. CM is a ray parallel to AB and l is a line which passes through E, F and intersects CM at D.
Since ED∥BC and BE∥CD, so BCDE is a parallelogram.
Now in △AEF and △CDF,
∠BAC=∠ACD (AB∥CD and AC is transversal, so alternate interior angles are equal)
AE=CD (Since AE=BE and BE=CD since BCDE is a parallelogram)
∠AEF=∠CDF (AB∥CD and DE is transversal, so alternate interior angles are equal)
So, △AEF≅△CDF (ASA Congruence Rule)
So, AF=CF (CPCT). Hence, proved.

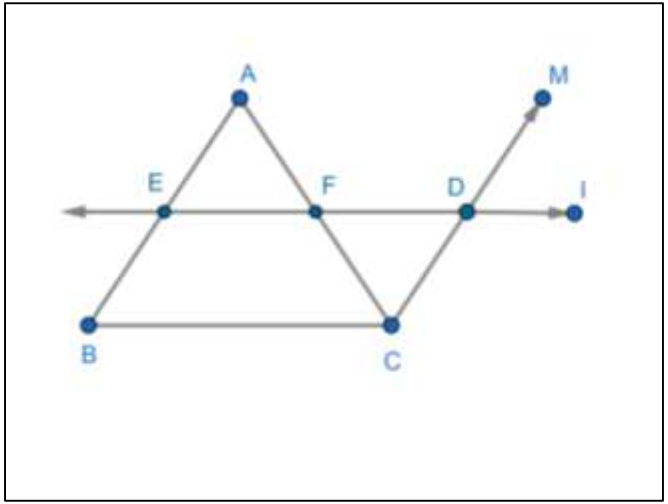

Fig. 1.13

Summary

- Sum of the angles of a quadrilateral is 360°.
- A diagonal of a parallelogram divides it into two congruent triangles.
- In a parallelogram,
  (i) opposite sides are equal (ii) opposite angles are equal
  (iii) diagonals bisect each other
- A quadrilateral is a parallelogram, if
  (i) opposite sides are equal or (ii) opposite angles are equal
  or (iii) diagonals bisect each other
  or (iv)a pair of opposite sides is equal and parallel

Summary (Contd..)

- Diagonals of a rectangle bisect each other and are equal and vice-versa.
- Diagonals of a rhombus bisect each other at right angles and vice-versa.
- Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
- The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
- A line through the mid-point of a side of a triangle parallel to another side bisects the third side.
- The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

THANK YOU